https://www.linkedin.com/feed/update/urn:li:activity:6480681624529965056

S-35. Proposed by R.S.Luthar, University of Wisconsin, Janesville.

Prove that

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2} > \cot A+\cot B+\cot C,$$

where A,B,C are angles of a triangle.

Solution by Arkady Alt, San Jose, California, USA.

Let $\triangle ABC$ be some triangle with sidelengths a,b,c, circumradius R, inradius r and semipetimeter s. Noting that $\cot\frac{X}{2}=\frac{s-x}{r}$, where $(X,x)\in\{(A,a),(B,b),(C,c)\}$ we obtain $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{s-a}{r}+\frac{s-b}{r}+\frac{s-c}{r}=\frac{s}{r}$ and, therefore,

$\sum\cot\frac{A}{2}>\sum\cot A \iff \frac{s}{r}>\sum\frac{\cos A}{\sin A} \iff \frac{s}{2Rr}>\sum\frac{\cos A}{2R\sin A} \iff$

$\frac{2s^2}{4Rrs}>\sum\frac{\cos A}{a} \iff \frac{2s^2}{abc}>\sum\frac{\cos A}{a} \iff 2s^2>\sum bc\cos A \iff$

$4s^2>2\sum bc\cos A=\sum(bc\cos A+ca\cos B) \iff$

$(a+b+c)^2>\sum c(b\cos A+a\cos B)=\sum c^2 \iff$

(1) $\quad 2ab+2bc+2ca-a^2-b^2-c^2>0.$

Since $a+b>c$ implies $\sqrt{a}+\sqrt{b}>\sqrt{c}$ and cyclic we have $\sqrt{b}+\sqrt{c}>\sqrt{a}$, $\sqrt{c}+\sqrt{a}>\sqrt{b}$ then $2ab+2bc+2ca-a^2-b^2-c^2=$

$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\left(\sqrt{a}+\sqrt{b}-\sqrt{c}\right)\left(\sqrt{b}+\sqrt{c}-\sqrt{a}\right)\left(\sqrt{c}+\sqrt{a}-\sqrt{b}\right)>0.$

S-36. Proposed by R.S.Luthar, University of Wisconsin, Janesville.

Eliminate u and v from the following set of equations:

$$x=\cos u+\cos v, y=\sin u-\sin v, z=\cos(u-v).$$

Solution by Arkady Alt, San Jose, California, USA.

We have $x^2+y^2=(\cos u+\cos v)^2+(\sin u-\sin v)^2=$
$2+2(\cos u\cos v-\sin u\sin v)=2+2\cos(u-v)=2+2z.$

S-37. Proposed by Mircea Ghita, Flushing, NY.

Solve the equation $\frac{1}{\sin^{2k}x}+\frac{1}{\cos^{2k}x}=8$, where k is a positive integer.

Solution by Arkady Alt, San Jose, California, USA.

In the case $k=1$ the equation becomes $\frac{1}{\sin^2x}+\frac{1}{\cos^2x}=8 \iff$
$1=8\sin^2x\cos^2x \iff 1=2\sin^2 2x \iff \sin 2x=\pm\frac{1}{\sqrt{2}} \iff x=\frac{\pi}{8}+\frac{n\pi}{2}, n\in\mathbb{Z}.$

Let now $k\geq 2$.

Applying inequality

(1) $\quad (a+b)^k\left(\frac{1}{a^k}+\frac{1}{b^k}\right)\geq 2^{k+1}, a,b>0$

(by AM-GM inequality $(a+b)^k\geq 2(ab)^{k/2}$ and $\frac{1}{a^k}+\frac{1}{b^k}\geq 2\cdot\frac{1}{a^{k/2}b^{k/2}}$)

to $(a,b)=(\sin^2x,\cos^2x)$ we obtain

$$8 = \frac{1}{\sin^{2k}x} + \frac{1}{\cos^{2k}x} = (\sin^2 x + \cos^2 x)^k \left(\frac{1}{\sin^{2k}x} + \frac{1}{\cos^{2k}x} \right) \geq 2^{k+1}$$

and that implies $k \leq 2$. Thus $k = 2$ and since equality in inequality (**1**) occurs iff $a = b$, that is in our case $\sin^2 x = \cos^2 x$, then

$$8 = \frac{1}{\sin^4 x} + \frac{1}{\cos^4 x} \Leftrightarrow \sin^2 x = \cos^2 x = \frac{1}{2} \Leftrightarrow x = \frac{\pi}{4} + n\pi, n \in \mathbb{Z}.$$

So, equation $\frac{1}{\sin^{2k}x} + \frac{1}{\cos^{2k}x} = 8$ have no solution if $k \geq 3$ and

$$x = \frac{\pi k}{8} + \frac{nk\pi}{2}, n \in \mathbb{Z}, k = 1, 2.$$